Optimal Trajectory Planning for the Apollo Moon Landing:

Descent, Ascent, and Aborts

Aero-Astro 16.323 Optimal Control
Final Project

Created By
Duncan Miller
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Massachusetts Institute of Technology
Cambridge, MA

Abstract

This paper presents optimal trajectory solutions for guiding the Apollo Lunar Excursion Module (LEM) to and from the Moon's surface. $\mathbb{GPOPS}-\mathbb{II}$ was used to solve the nonlinear control problem, providing full state and control histories for optimal ascent, descent and abort scenarios. Although historically Apollo did not follow an optimal trajectory due to terrain and navigation constraints, the methods presented here can be used by future mission planners to quantify mission risk and potentially adapt the concept of operations for optimal astronaut safety.

Contents

Duncan Miller, *16.323 Student*
duncanlm@mit.edu

1 Apollo Overview

The highlight of the Apollo space program was mankind's repeated landings on the surface of the Moon. To this end, the Lunar Excursion Module (LEM) was designed to carry astronauts to and from the lunar surface. The LEM provided man's first aerial approach into such an alien environment. Although trajectories were simulated many times preflight, the high terrain variability, guidance error, sensor noise, and visibility hazards added extremely high risk to the mission profile. Lunar injection and subsequent re-rendezvous phases were the most complicated control sequences of the Apollo mission and are the focus of this project.

The Apollo mission trajectory was not optimal in the traditional sense of time or fuel. NASA engineers instead optimized the probability of mission success and astronaut safety, given the constraints on computational power and guidance instruments at the time. Moreover, the variations in lunar terrain were only known a priori within about $\pm$ 20,000 ft - even the lunar reference radius from which the terrain is measured had large uncertainties. Also, worst case instrument readings necessitated larger margins on state estimates. To account for these error dispersions, the Apollo mission profiles were largely conservative and even relied on manned piloting during final descent. This added an additional constraint on the flight path angle so that the crew could use the window to evaluate the landing site and correct as needed. As a result, the final trajectories for descent, ascent and abort deviated from the fuel and time optimal solutions presented here.

2 Project Scope

For this project, I have developed a three degree of freedom simulation of the Apollo Lunar Lander. Specifically, we explore three optimal control problems:

1. The descent from lunar orbit to a soft landing;
2. The ascent to lunar orbit to rendezvous with the Lunar Orbiter; and
3. The ascent and rendezvous from an aborted landing.

This project is of intellectual merit for three reasons. First, it is relevant to the class material covered this semester. I first researched the 6DOF equations of motion for an orbiting satellite. Then I formulated the objective functions, boundary conditions and physical constraints. I examined the vehicle response and was able to draw conclusions for general OCPs. Second, we can use the results from this analysis to learn from and critically analyze the Apollo trajectory design, especially for abort cases. Finally, this paper collects existing literature on the subject of manned lunar missions and creatively investigates an original approach to the abort planning policy. The methods presented herein may be scaled and adapted to future manned space missions to the Moon and Mars, which I hope to be a part of.

This project delivers (1) a detailed discussion on the problem formulation including a simplified vehicle model that can relate control inputs to vehicle accelerations, (2) results and conclusions from a variety of descent, ascent and abort scenarios, (3) a body of Matlab code, GPOPS config files, plots and a simulation generated for this assignment.

3 Dynamics

The equations of motion of a satellite orbiting a massive body like the moon are well known. For the scope of this project, the spacecraft has been reduced to a point mass and its gravitational effect on the moon-satellite system has been omitted. The Moon itself has been modeled as a perfectly spherical body and additional gravitational terms from the Earth and the Sun have been neglected. Under these assumptions, the orbit equations can be realized in following spherical coordinates [6].

$$\begin{cases} \dot{r} &= v_r \\ \dot{\theta} &= \frac{v_t}{r\cos\phi} \\ \dot{\phi} &= \frac{v_n}{r} \\ \dot{v_r} &= -\frac{\mu}{r^2} + \frac{v_t^2+v_n^2}{r} + \frac{T}{m}\sin\alpha\cos\beta \\ \dot{v_t} &= \frac{v_t}{r}(v_n\tan\phi - v_r) + \frac{T}{m}\cos\alpha\cos\beta \\ \dot{v_n} &= -\frac{v_t^2}{r}\tan\phi - \frac{v_r v_n}{r} + \frac{T}{m}\sin\beta \\ \dot{m} &= -\frac{T}{I_{sp}g_0} \end{cases} \tag{1}$$

Here we have defined seven state variables. r represents the radial distance from the center of the moon to the Lunar Excursion Module. θ is the true anomaly of the spacecraft, the angular position of its orbit. Since the moon has been modeled as a perfect sphere, and our problem scope only includes the Moon's sphere of influence since lunar insertion, it is natural to define the travel angle of the spacecraft from $\theta(t_0) = 0$ to $\theta(t)$. ϕ has been defined as the declination of the orbit plane from Moon-Centered Inertial coordinates. v_r denotes the in-plane radial velocity and v_t denotes the in-plane tangential velocity of the spacraft. v_n represents the out-of plane tangential velocity of the LEM.

Finally, we have introduced a seventh state variable to represent the mass of the spacecraft. This is important to include because as the spacecraft uses fuel, the thrust-to-weight of the vehicle increases dramatically. Indeed, for a vehicle made up of 50% fuel, the thrust-to-weight would double by the end of its mission life. Thus we have introduced a time varying state variable to account for this.

There are three control variables specified in this problem formulation: thrust magnitude (T), in-plane thrust angle (α) and out-of-plane thrust angle (β). Spacecraft attitude has not been included in our controllable state set. The Apollo Lunar Module used a throttle-able, trim gimballed descent propulsion system (DPS) and ascent propulsion system (APS). The DPS and APS engines were directed in the minus-Z body frames and attitude was control separately via the hypergolic thrusters of the Reaction Control System (RCS). In this problem formulation, the assumption is that the RCS does its job of orienting the vehicle to direct the thrust vector according to α and β. It would be interesting to see the two problems coupled together (since the RCS thrusters do affect the trajectory to some degree) but this is left for future work.

We are able to reduce our state space, equations of emotion, and control variables significantly by guessing an optimal solution to the problem. Since gravitational forces only act in the plane of the orbit, and pseudo acceleration terms from a rotation reference frame do not induce out of plane dyanmics, we can assume a planar solution. This simplifies our model as follows.

$$
\begin{cases}
\dot{r} &= v_r \\
\dot{\theta} &= \frac{v_t}{r\cancelto{1}{\cos\phi}} \\
\cancelto{0}{\dot{\phi}} &= \frac{\cancelto{0}{v_n}}{r} \\
\dot{v}_r &= -\frac{\mu}{r^2} + \frac{v_t^2 + \cancelto{0}{v_n^2}}{r} + \frac{T}{m}\sin\alpha\cancelto{1}{\cos\beta} \\
\dot{v}_t &= \frac{v_t}{r}\left(\cancelto{0}{v_n\tan\phi} - v_r\right) + \frac{T}{m}\cos\alpha\cancelto{1}{\cos\beta} \\
\cancelto{0}{\dot{v}_n} &= -\frac{v_t^2}{r}\tan\phi - \frac{\cancel{v_r v_n}}{r} + \frac{T}{m}\sin\beta \\
\dot{m} &= -\frac{T}{I_{sp}g_0}
\end{cases}
\tag{2}
$$

We have cleaned up the terms and used the following dynamics in our nonlinear solver.

$$
\boxed{
\begin{aligned}
\dot{r} &= v_r && (3)\\
\dot{\theta} &= \frac{v_t}{r} && (4)\\
\dot{v}_r &= -\frac{\mu}{r^2} + \frac{v_t^2}{r} + \frac{T}{m}\sin\alpha && (5)\\
\dot{v}_t &= -\frac{v_t v_r}{r} + \frac{T}{m}\cos\alpha && (6)\\
\dot{m} &= -\frac{T}{I_{sp}g_0} && (7)
\end{aligned}
}
$$

Next I have defined the constants used for this solution. Many of the parameters related to the design properties of the LEM are referenced from [5]. The ascent stage mass was taken from the National Space Science Data Center [1].

$$
\begin{aligned}
\text{Moon's Gravitational Constant:} \quad & \mu = 4902.9 km^3/s^2 \\
\text{Moon's Gravitational Acceleration:} \quad & g = 1.622 \times 10^{-3}\,\mathrm{km/s^2} \\
\text{Moon Radius:} \quad & R_M = 1738\,\mathrm{km} \\
\text{Earth's Gravitational Acceleration:} \quad & g_0 = 9.81 \times 10^{-3}\,\mathrm{m/s^2} \\
\text{CSM Parking Orbit Radius} \quad & h_p = 111\,\mathrm{km} \\
\text{Orbital Parking Velocity:} \quad & v_{p_t} = \sqrt{\frac{\mu}{R_M + h_p}} = 1.628\,\mathrm{km/s} \qquad (8) \\
\text{Specific Impulse (ascent \& descent):} \quad & I_{sp} = 311\,\mathrm{s} \\
\text{Total Mass of Descent Stage:} \quad & m_{t_{ds}} = 10\,465\,\mathrm{kg} \\
\text{Propellant Mass of Descent Stage:} \quad & m_{p_{ds}} = 8355\,\mathrm{kg} \\
\text{Total Mass of Ascent Stage:} \quad & m_{t_{as}} = 4774\,\mathrm{kg} \\
\text{Propellant Mass of Ascent Stage:} \quad & m_{p_{as}} = 2615\,\mathrm{kg}
\end{aligned}
$$

4 Gauss Pseudospectral Method

The general procedure for solving Optimal Control Problems (OCPs) has been summarized in Figure 1. First, the optimal control problem is formulated and subsequently discretized for a nonlinear numerical solver to optimize. Once a solution is found (or not found), the mesh may be adapted and the problem may be retranscribed as needed to converge to an extremal.

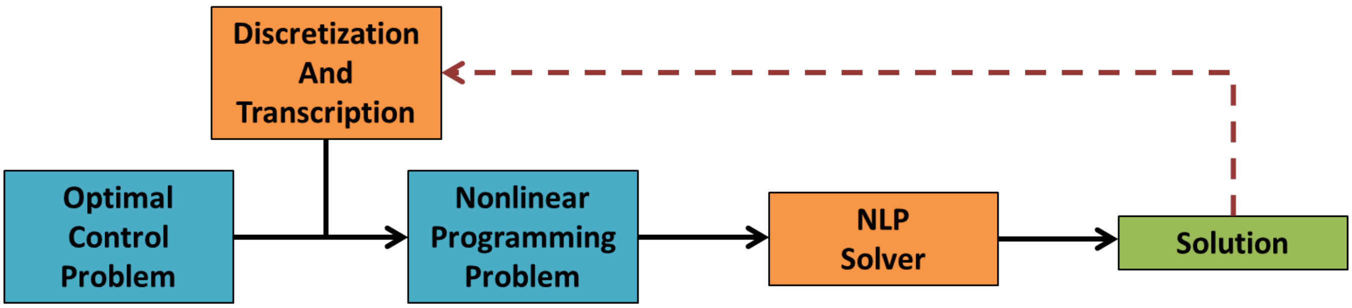

Figure 1: Traditional Optimal Control Iteration Loop

The optimal control problems presented in this paper can be written in the followin general form, with the objective function, dynamic constraints and boundary conditions given as J, $\frac{d\mathbf{x}}{dt}$ and ϕ respectively.

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & J = \mathbf{\Phi}(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(t), t)dt \\ \text{subject to} \quad & \frac{d\mathbf{x}}{dt} = f(\mathbf{x}(t), \mathbf{u}(t), t) \\ & \phi(\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f) = 0 \end{aligned} \tag{9}$$

For this paper, I have taken the OCPs and solved them using a recently developed transcription method called the Gauss pseudospectral method ($\mathbb{GPOPS} - \mathbb{II}$). Although detailing the methodology of $\mathbb{GPOPS} - \mathbb{II}$ is beyond the scope of this paper, we can make some observations here that will assist in our interpretation of the forthcoming solutions presented. The reader is referred to [2] for a more in depth explanation of pseudospectral methods.

$\mathbb{GPOPS} - \mathbb{II}$ has been designed to solve nonlinear, continuous, and multi-phase problems. $\mathbb{GPOPS} - \mathbb{II}$ uses Gaussian quadrature method to solve the OCP by obtaining a sparse, finite nonlinear programming problem (NLP) before translating it for existing solvers (like SNOPT, IPOPT) to evaluate. The important realization is that $\mathbb{GPOPS} - \mathbb{II}$ iterates and approximates the solution using adaptive mesh refinement until the desired accuracy is achieved. The adapted Figure 2 summarizes the key parameters and shows how collocation points may be much denser in some regions than other. We observe this phenomenon in later abort cases, as shown in Figure 13.

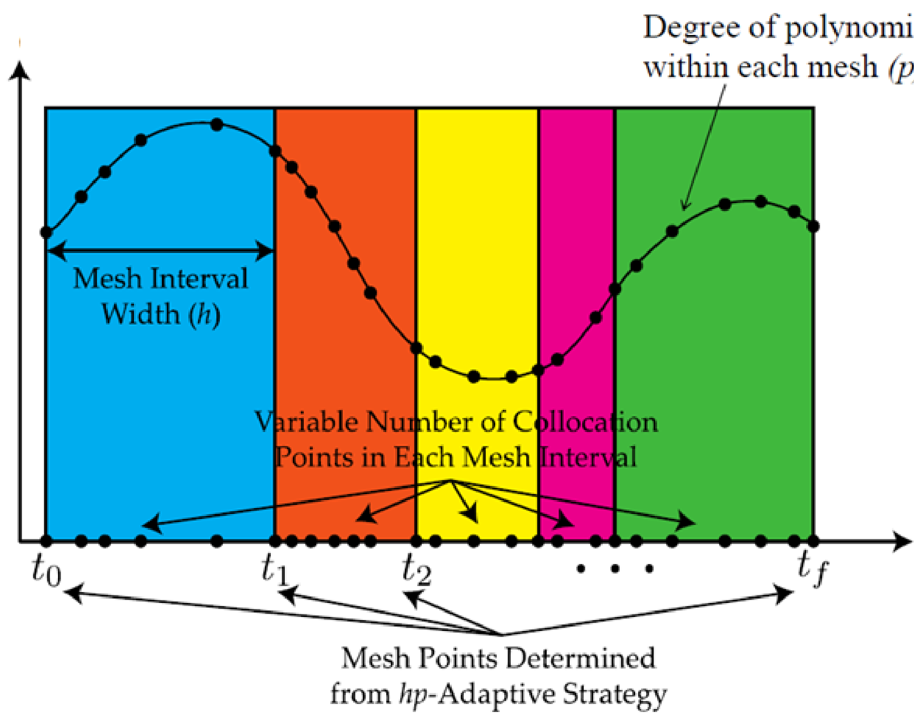

Figure 2: Adapted Figure Detailing the Meshing Algorithm for GPOPS-II

Thus, $\mathbb{GPOPS} - \mathbb{II}$ has been selected as the solver for three primary reasons. It employs first and second sparse finite-differencing to find derivatives, uses an hp-adaptive mesh refinement method (ideal for near discontinuities), and is rather simple for problem formulation (only requiring Matlab).

5 The Descent Problem

5.1 Apollo Flight Plan

First we consider the problem of landing the Apollo astronauts on the surface of the Moon from a lunar parking orbit. In theory, the optimal control maneuver would use two impulsive thrusts to complete Hohman transfer orbit from the dark side of the moon to the landing zone. However, the descent stage is limited to a maximum thrust to weight ratio of 1.8 (at landing), which cannot be approximated as impulsive.

The final Apollo trajectory settled on a two phase design: (1) separation from the Command and Service Module (CSM) followed by an obit transfer from the 50 nautical mile parking orbit (111km), (2) braking phase beginning at the 50,000 ft pericynthion, (3) final landing phase. This was implemented, as previously discussed, due to state estimation uncertainties, high lunar terrain, and the high flight path angle requirement (for pilot viewing through the window).

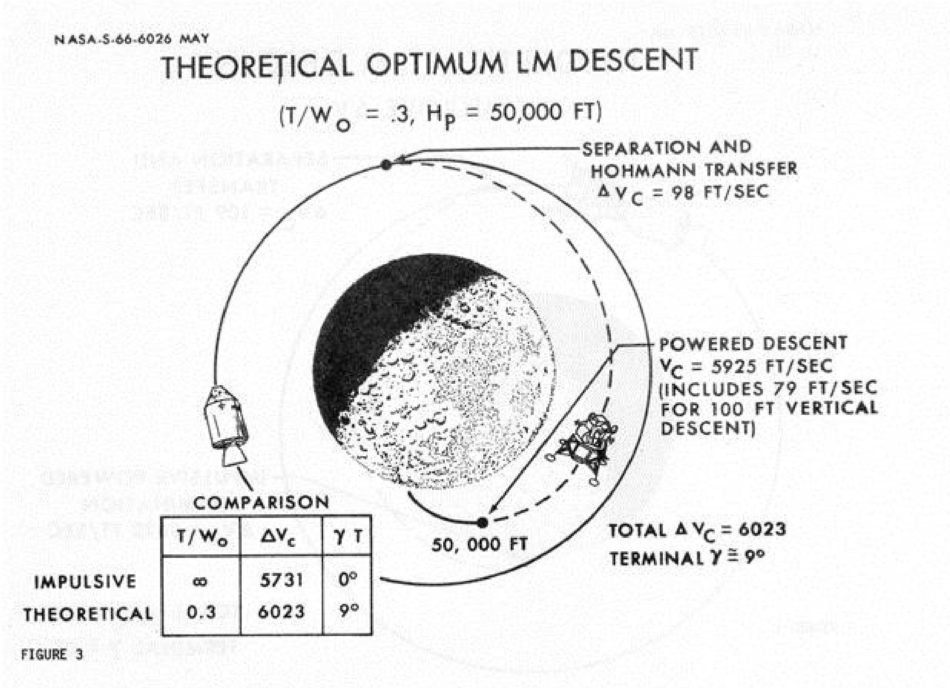

Figure 3: Actual Descent Trajectory Used for Apollo Missions [3]

5.2 Minimum Time Problem Formulation

First, we consider the minimum time descent problem for the LEM from a 111 km parking orbit. Using the final time as the objective function is reasonable, considering the astronauts have a fixed amount of oxygen available for the mission. Additionaly, the astronauts would like to spend as much time as possible on the lunar surface conducting experiments. Thus, within the fuel constraints (and some margin to account for uncertainties), a minimum time problem is formulated. We will see that this choice actually keeps the propellant mass high, close to the optimal fuel efficient path presented later.

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & J = t_f \\ \text{subject to} \quad & \dot{x} = f(x,u) \\ & r > R_M \\ & (m_{t_{ds}} + m_{t_{as}} - m_{p_{ds}}) \leq m \leq m_{t_{ds}} + m_{t_{as}} \\ & T : \{0, [0.1T_{max}, 0.6T_{max}], T_{max}\} \end{aligned} \tag{10}$$

During descent, the LEM must carry both the descent and ascent stages. Morover, the thrust is allowed to be throttleable, from 10% to 60% or full throttle at $T_{max} = 45.03kN$. Of course, the Moon's surface is modeled as flat and impenetrable.

We define the boundary conditions as follows. Initially, the LEM begins in a circular orbit at 11 km with full propellant.

$$r_0 = r_M + h_p \qquad \theta_0 = 0 \qquad v_{r_0} = 0 \qquad v_{t_0} = \sqrt{\frac{\mu}{r_0}} \qquad m_0 = m_{t_{ds}} + m_{t_{as}} \tag{11}$$

The spacecraft must touch down on the surface with zero velocity (a 'soft landing').

$$r_f = r_M \qquad v_{r_f} = 0 \qquad v_{t_f} = 0 \tag{12}$$

The parking orbit has been modeled as circular, which is a resonable assumption. The target orbits for rendezvous are naturally circular. This ensures that any rendezvousing satellite can match both position and velocity during approach phasing. We also note that θ_f is unconstrained. Although the actual Apollo mission had a specific landing region in mind, it would be trivial to propagate backwards from t_f and determing when descent should begin in order to hit the landing zone (assuming perfect state knowledge).

5.3 Results

The problem was input into $\mathbb{GPOPS}-\mathbb{II}$ and solved numerically. Some parameters were tuned, such as the mesh, initial guess, and control bounds so that the solver converged to an optimal trajectory (Figure 4, Figure 5).

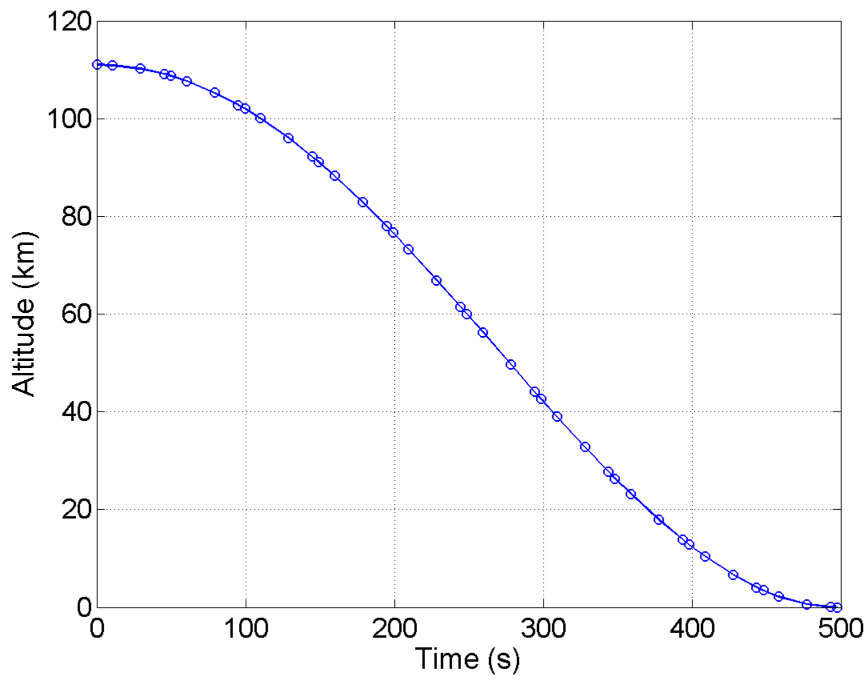

Figure 4: Altitude Trajectory for Optimal Lunar Descent from 111km

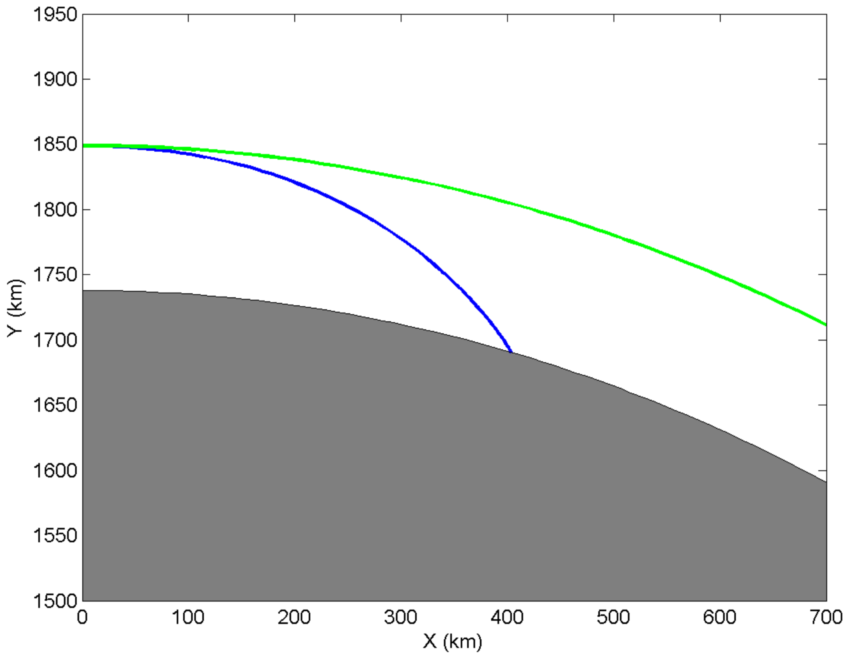

Figure 5: Planar Trajectory in MCI Coordinates for Optimal Lunar Descent from 111 km

The green path is the propagated CSM trajectory. The full state histories for the LEM follow the paths summarized in Figure 6.

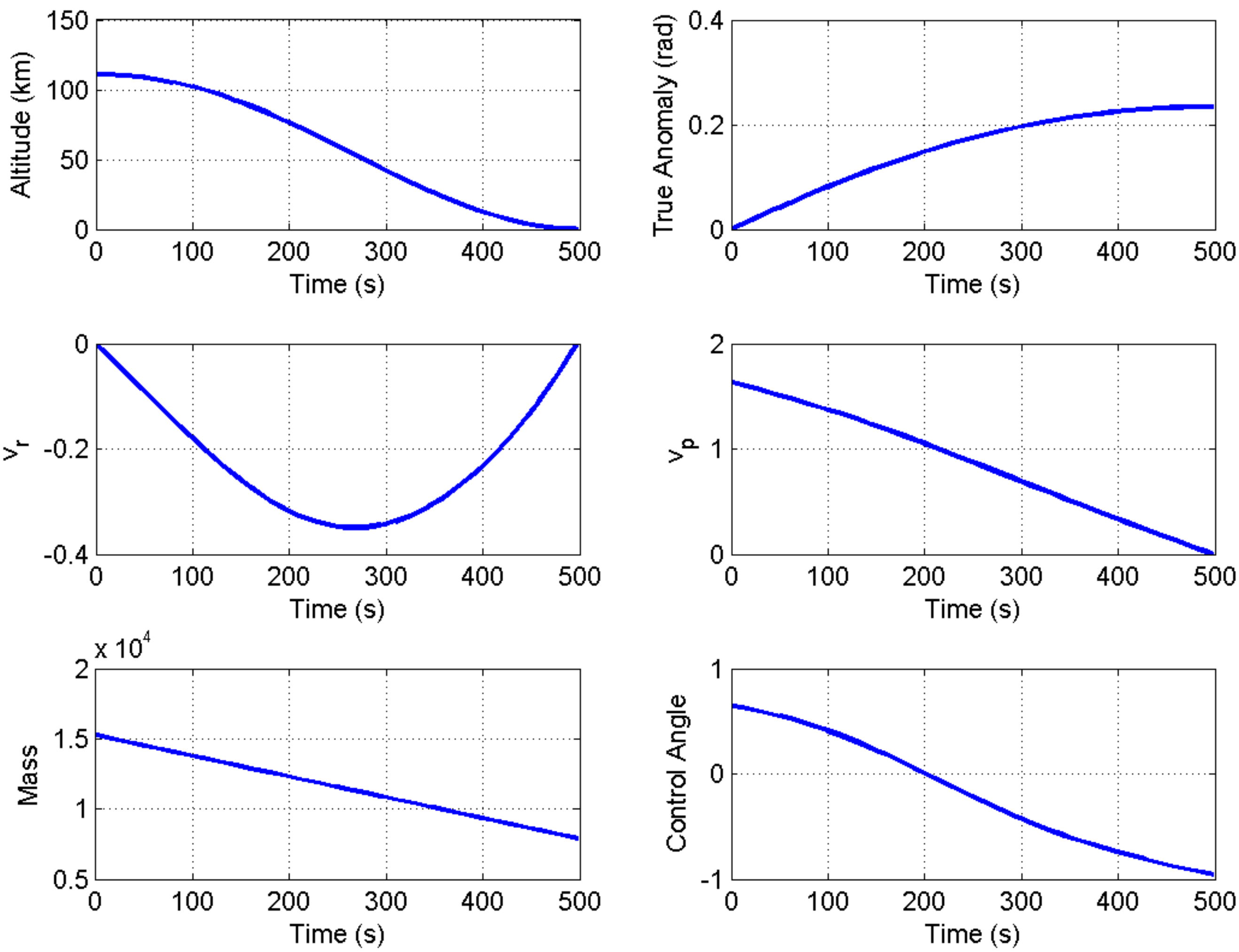

Figure 6: Summarized State and Control Histories for the Lunar Descent

As expected, all states meet the boundary conditions specified. Although the thrust was allowed to be throttleable, the maximum thrust was used throughout the trajectory (indicated by the linear negative slope for mass). However, the control angle varried continuously, first thrusting forward to slow orbit velocity and then thrusting down for a soft landing.

While implementing the solver, I also adjusted various parameters to observe their effects on orbit trajectories. For example, if the parking orbit was instead 5km, we obtain the results shown in Figure 7.

5.4 5km Parking Orbit

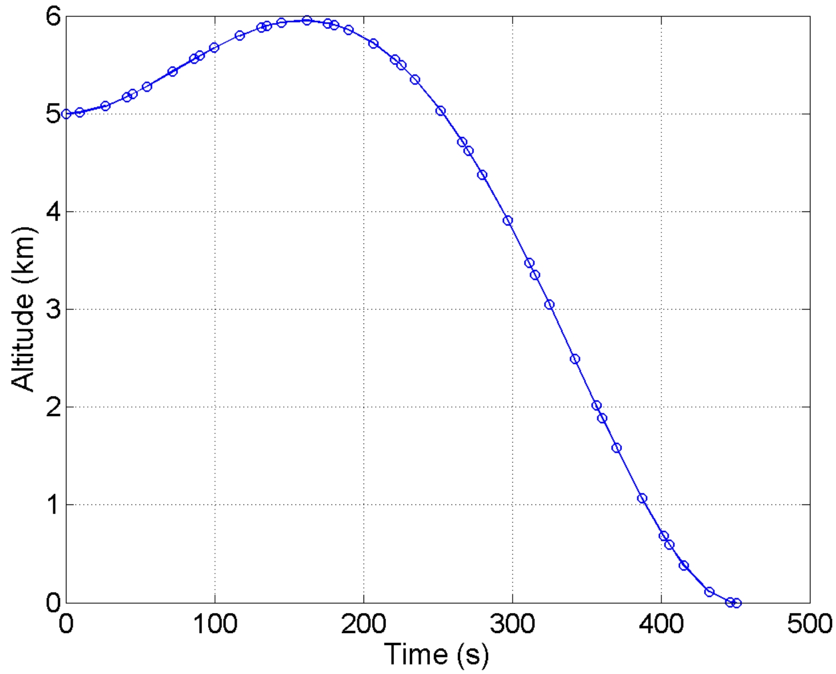

Figure 7: Altitude Trajectory for Optimal Lunar Descent from 5km

It is interesting to note that if we started from a 5km altitude, the optimal trajectory in fact first raises the orbit before falling to the surface. This is attributed to the limited control authority of the descent thruster. At a lower altitude, the orbit velocity is very high. Thus, by slowing down, the altitude increases temporarily as the spacecraft transitions to an elliptic orbit with higher apoapsis. If given a higher thrust, I was able to show that the minimum time trajectory followed a path similar to Figure 4.

5.5 Minimum Mass Trajectory

We also consider the minimum fuel descent trajectory. The objective funciton can be adjusted as follows.

$$\underset{u}{\text{minimize}} \quad J = m_f \tag{13}$$

As expected, Figure 8 shows the time to land is significantly increased by a factor greater than 3. Since the descent stage is throttleable, the optimal control history hovers around 20% for the majority of the journey. This is much closer the actual flight profile implemented in Apollo. Indeed the descent stage was first throttled to 10% to begin descent for nominal engine checkout and minimal fuel. More details can be found in the Appendix, Figure 24.

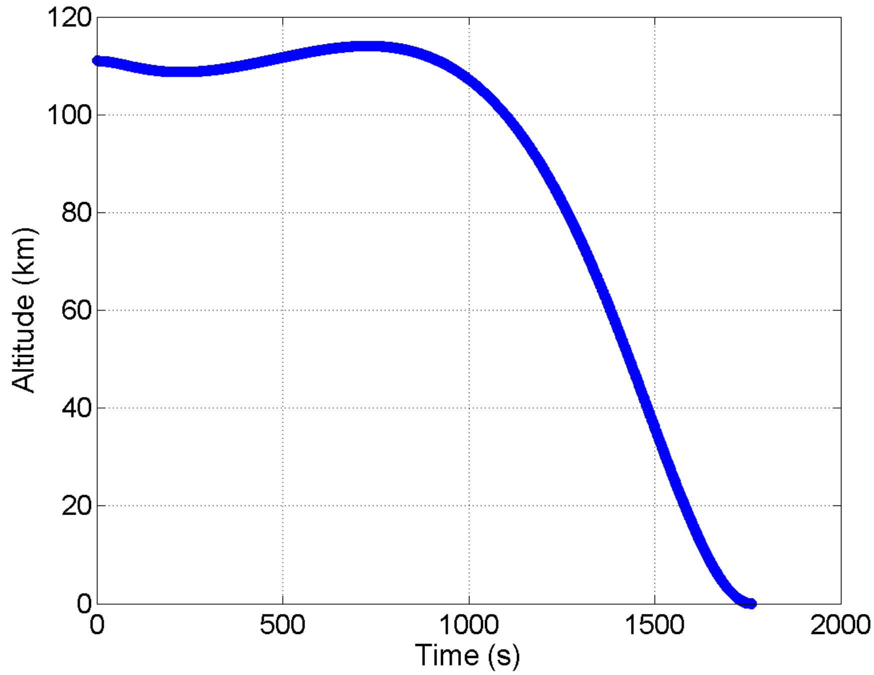

Figure 8: Altitude Trajectory for Minimum Mass Lunar Descent

6 The Ascent Problem

6.1 Apollo Flight Plan

The Apollo flight plan followed two phases: vertical rise and orbit insertion. The vertical rise was necessary to clear the nearby lunar terrain prior to ‘pitch-over’ and subsequent orbit insertion. First, the astronauts launched from the surface into a 17 x 83 km orbit. The orbit was then circularized into a 83 km orbit, and subsequently inserted themselves into an orbit with a 28 km height differential [10]. This was known as the coelliptic method as it gave the ascent stage a large buffer to rendezvous with the CSM (about 3.5 hours or two orbits).

Compared to the descent problem, optimal ascent was considered lower risk and did not require use of the landing radar. Also since the ascent engine was non-throttleable and non-gimbaled, the attitude was controlled soley by the RCS.

6.2 Problem Formulation

The ascent problem can be modeled very similarly to the descent problem with updated boundary conditions. Again, I have taken the objective function to be minimum time. Presumably, there is a fixed amount of oxygen and supplies and the NASA engineers would like to maximize astronaut time on the surface (with propellant margin).

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & J = t_f \\ \text{subject to} \quad & \dot{x} = f(x,u) \\ & r > R_M \\ & (m_{t_{as}} - m_{p_{as}}) \leq m \leq m_{t_{as}} \\ & T : \{0, T_{max}\} \end{aligned} \tag{14}$$

Again, the engine is non throttleable so the thrust is limited to either off or full throttle ($T_{max} = 16.0kN$). The boundary conditions are given as follows. The initial conditions are:

$$r_0 = r_M \qquad \theta_0 = 0 \qquad v_{r_0} = 0 \qquad v_{t_0} = 0 \qquad m_0 = m_{t_{as}} \tag{15}$$

We note that the mass now only includes the ascent stage mass. The descent stage is abandoned on the surface. The final conditions are:

$$r_f = r_M + h_p \qquad \theta_f = \frac{v_p}{r} * \Delta t \qquad v_{r_f} = 0 \qquad v_{t_f} = \sqrt{\frac{\mu}{r_f}} \tag{16}$$

The new final condition on θ comes from the need to rendezvous exactly with the Command and Service Module. For a nominal ascent, the astronauts are instructed to hold on the surface of the moon until the CSM comes into position, exactly Δt seconds prior to the scheduled rendezvous.

6.3 Results

In this section we present the solutions found for optimal ascent. I note that the definition of the control angle was reversed 180 degrees. This way, the thrust angle α stayed continuous throughout the entire trajectory, and did not have a discontinuous switch when crossing over the ± 180 degree boundary constraint.

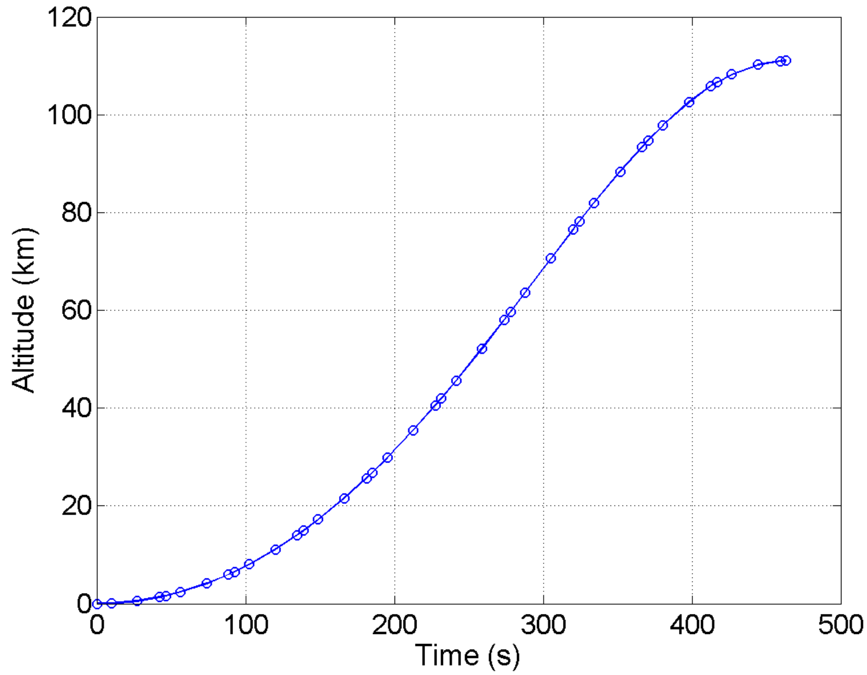

Figure 9: Altitude Trajectory for Optimal Lunar Ascent

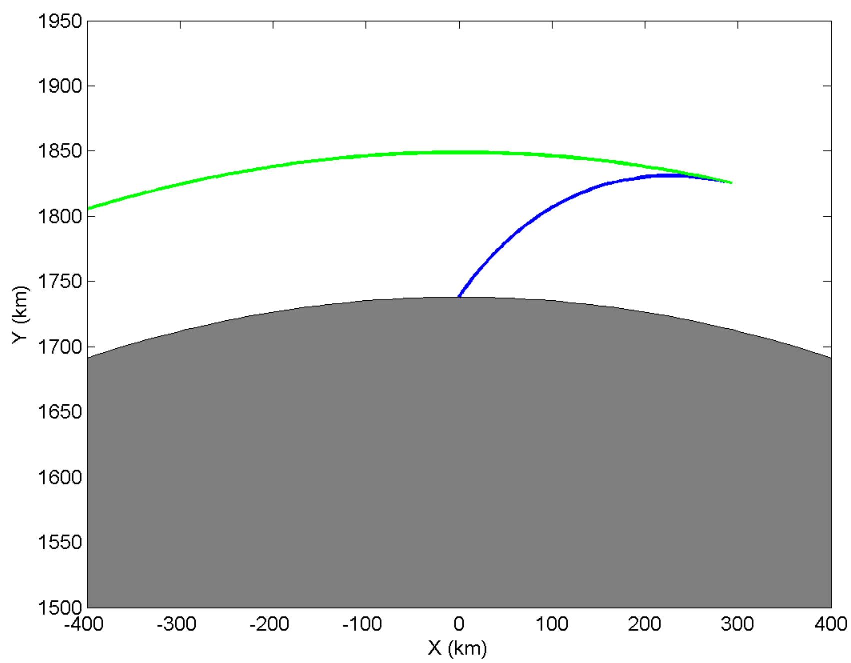

Figure 10: Moon Centered Inertial Trajectory for Optimal Ascent

Figure 11 details the full state history for this maneuver. Again, the thrust is held constant at maximum throttle for the duration of the ascent.

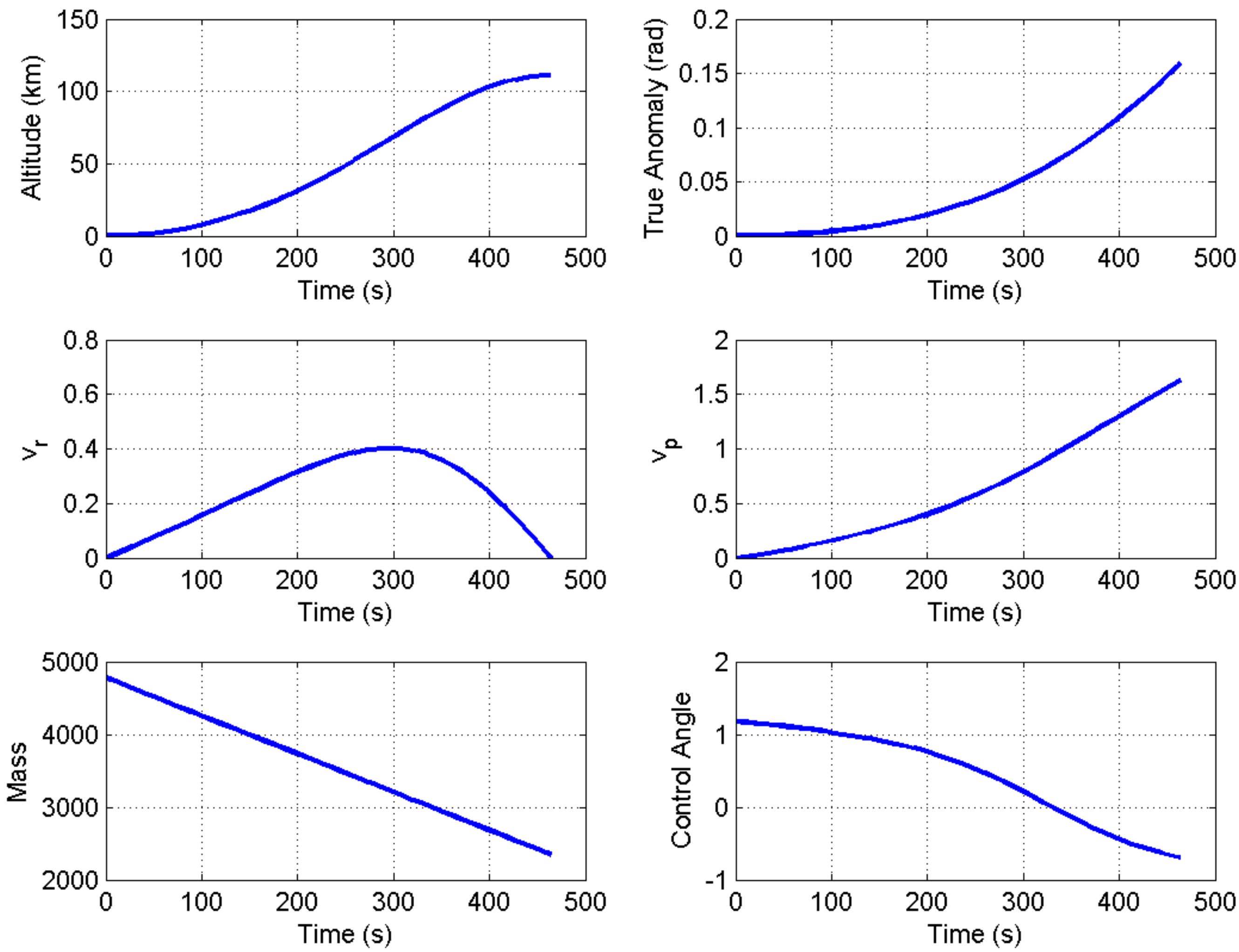

Figure 11: Summarized State and Control Histories for the Lunar Ascent

As in the descent problem, the optimal solution follows the expected smooth path. The thrust is constand and at full throttle for the duration of the trip (linear $\dot{m}$). We observe that all boundary conditions previously specified are satisfied. The flight time is slightly less than for optimal descent. We can attribute this to the higher thrust to weight ratio of the ascent engine. We notice that the thrust angle adjusts during the flight; it begins mostly downward to hoist the stage off of the surface. As the stage approaches orbital velocity, the LEM tips over and directs the thrust in the tangential direction.

7 The Abort Problem

7.1 Apollo Flight Plan

It was an Apollo mission requirement that the crew retain the capability to abort the lunar landing at nearly every phase of the descent trajectory. Only during the final powered braking, once committed to touchdown, was a direct abort not an option [10]. At any point in time, the commanding astronaut had two buttons at the ready: one that read 'Abort' and another that read 'Abort Stage'. The 'Abort' button was only able to be used early in the powered descent and actually used the descent engine to re-rendezvous with the CSM. The 'Abort Stage' was used in the case of a descent stage failure or if already deep into the descent maneuver. This enabled explosive charges to jettison the descent stage and re-rendezvous using only the ascent stage.

Fortunately, an abort was never commanded during the descent phase of any Apollo mission. However, there were contingencies in place just in case. As detailed in [8], rule 28-18 indicates that wherever possible, the abort will be initiated without stage separation. This ensured maximum propellant margin for re-rendezvous. Moreover, the abort guidance computer was computationally limited, preventing real-time planning of all possible abort options. In abort cases, it would have likely been necessary for computationally expensive calculations to be performed in Houston and relayed to the crew.

7.2 Solution Overview

Unlike the descent and ascent problems, an abort from powered descent may occur at any time - thus none of the initial state variables are known a priori. Unfortunately, due to physical constraints of the vehicle and mission, no single problem formulation can span the entire solution space.

As summarized in Figure 12, I have designed four possible abort scenarios that enable the astronauts to abort from powered descent at any point and re-rendezvous with the CSM.

- The 'pull-up' abort is feasible early in the descent and has the LEM immediately increase altitude and catch up with the CSM. The descent stage may or may not be jettisoned immediately.
- The 'two-staged' abort lengthens the window to immediately catch up with the CSM by first depleting the propellant in the descent stage before lighting the ascent stage and re-rendezvousing.
- The 'deflection' abort falls down to the Moon, maintaining a low orbit period and high speed before ascending to the parking orbit exactly with the CSM.
- The 'hop' abort occurs directly after touchdown and ensures the astronauts get off the surface as quickly as possible, regardless of CSM positioning.

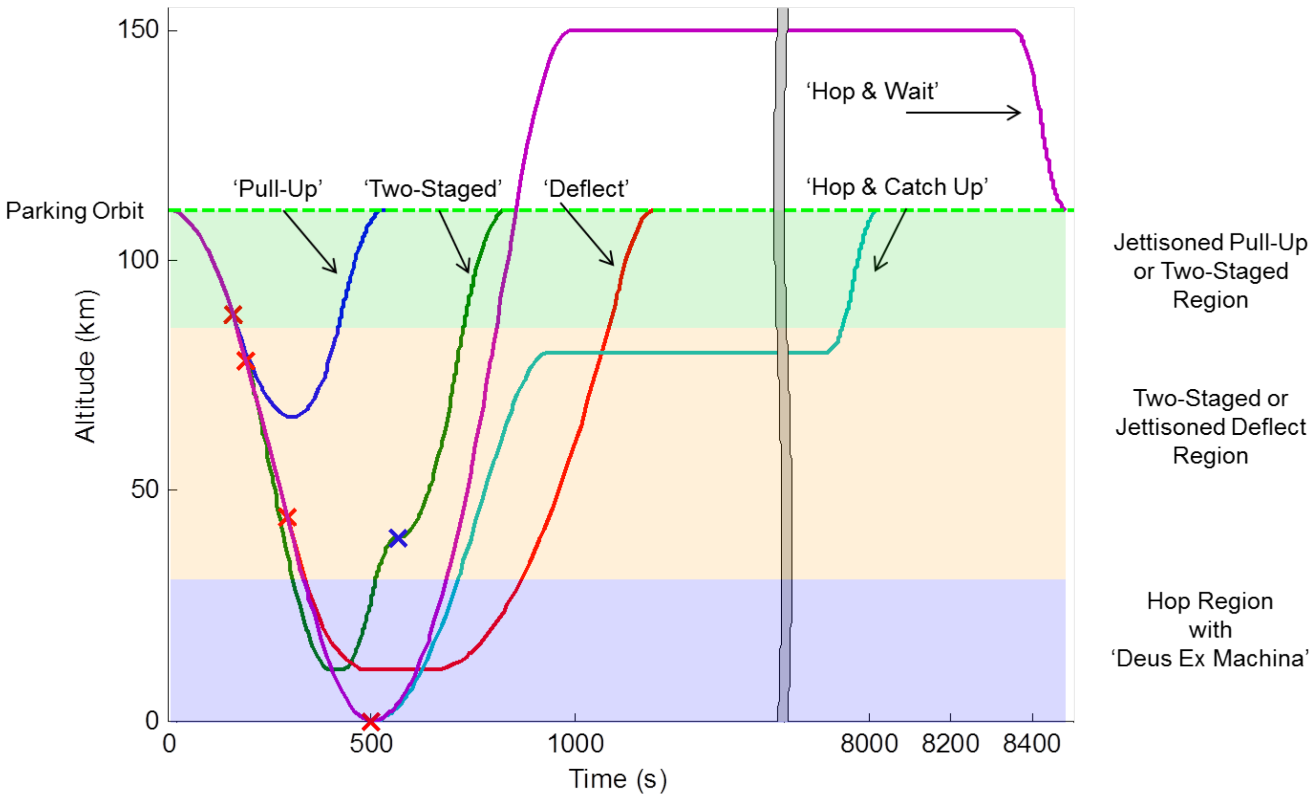

Figure 12: Proposed Solution Space for All Abort Cases

7.3 'Pull-Up' Problem Formulation

If the abort is initiated early in the descent trajectory, the altitude loss during the abort maneuver will be rather small. Although the LEM will traditionally begin descent slightly ahead of the CSM, the loss of tangential velocity will cause the CSM to pull ahead in the orbit. An early abort will allow the ascent stage to pull-up from the descent and catch up with the CSM.

The problem has been divided into two phases. First is the optimal descent phase as derived in a previous section. A new phase has been initiated at any arbitrary time after descent (down to about 80 km). The new additional phase is formulated as follows.

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & J = t_f \\ \text{subject to} \quad & \dot{x} = f(x,u) \\ & r > R_M \\ & (m_{t_{as}} - m_{p_{as}}) \leq m \leq m_{t_{as}} \\ & T : \{T_{max}\} \end{aligned} \tag{17}$$

The problem has been formulated assuming the astronaut has selected 'Stage Abort', immediately jettisoning the descent stage. The traditional 'Abort' using just the descent stage is formulated exactly the same with the descent stage mass and thrust values instead.

Again, the engine is non throttleable (fixed at $T_{max} = 16.0kN$) and (in this case) need not be restartable. The boundary conditions for the abort phases are given as follows. The initial conditions for the pull-up phase are that the interior points between optimal descent and abort are continuous:

$$t_f^1 = t_0^2 \tag{18}$$
$$r(t_f^{(1)}) = r(t_0^{(2)}) \tag{19}$$
$$\theta(t_f^{(1)}) = \theta(t_0^{(2)}) \tag{20}$$
$$v_r(t_f^{(1)}) = v_r(t_0^{(2)}) \tag{21}$$
$$v_t(t_f^{(1)}) = v_t(t_0^{(2)}) \tag{22}$$
$$m_0 = m_{t_{as}} \tag{23}$$

We note that the mass now only includes the ascent stage mass because in this pull-up maneuver the descent stage has been jettisoned at the start of this phase (though it need not be in general). The final conditions are:

$$r_f = r_M + h_p \qquad \theta_f = \frac{v_p}{r} * \Delta t \qquad v_{r_f} = 0 \qquad v_{t_f} = \sqrt{\frac{\mu}{r_f}} \tag{24}$$

The final condition on θ ensures that the ascent stage catches up and rendevous exactly with the CSM. It should be noted that in the actual Apollo mission, the LEM began its descent slightly ahead of the CSM. However, the phasing difference between the two is relatively minor and really only reduces the Δt allowable for a successful pull-up by a small margin.

7.4 'Pull-Up' Results

The pull-up solution presented here has an abort initated at 150 seconds into the descent at an altitude of 88 km. This was approximately what I am calling the 'Point of No Return' to perform a pull-up abort with a jettisoned descent stage. Waiting longer than 150 seconds into the optimal descent left the CSM too far ahead for the ascent stage to rendezvous with directly. For a pull-up abort without stage separation, I would expect this Point of No Return to be even earlier in the descent considering the descent stage will have already consumed fuel during the first phase.

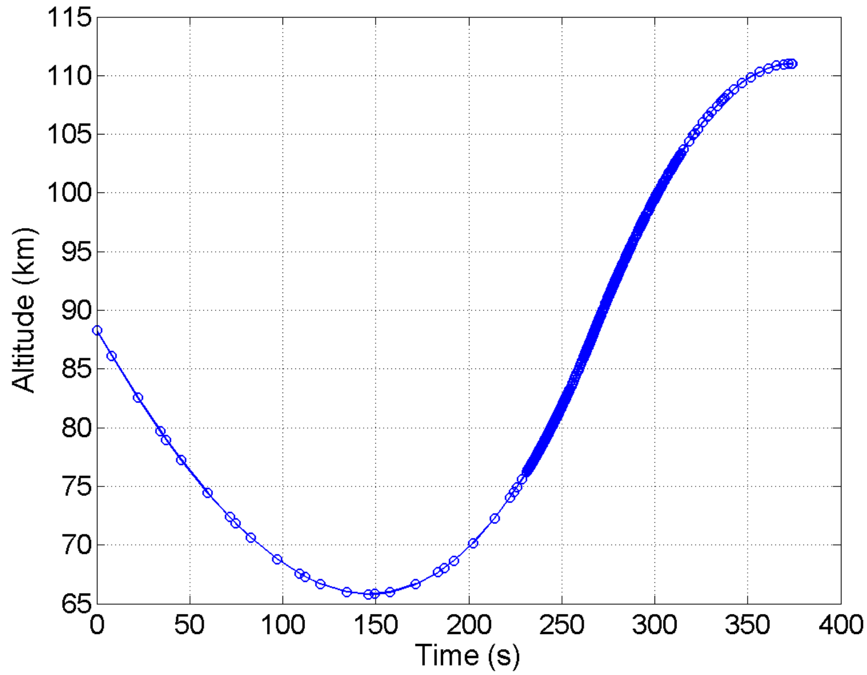

Figure 13: Altitude Trajectory for Optimal Lunar Abort

Here we present the trajectory plots for this sample pull-up abort. Figure 15 shows the path in the Moon Centered Inertial frame. The green represents the CSM maintaining a 111 km parking orbit; the blue shows the ascent stage path; the red shows the ballistic trajectory of the jettisoned descent stage.

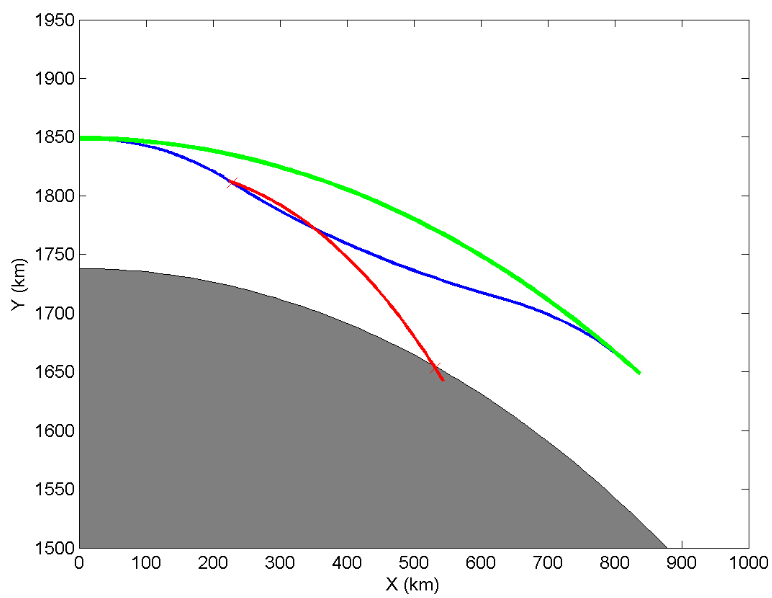

Figure 14: Planar Trajectory in MCI Coordinates for Optimal Pull-Up Abort

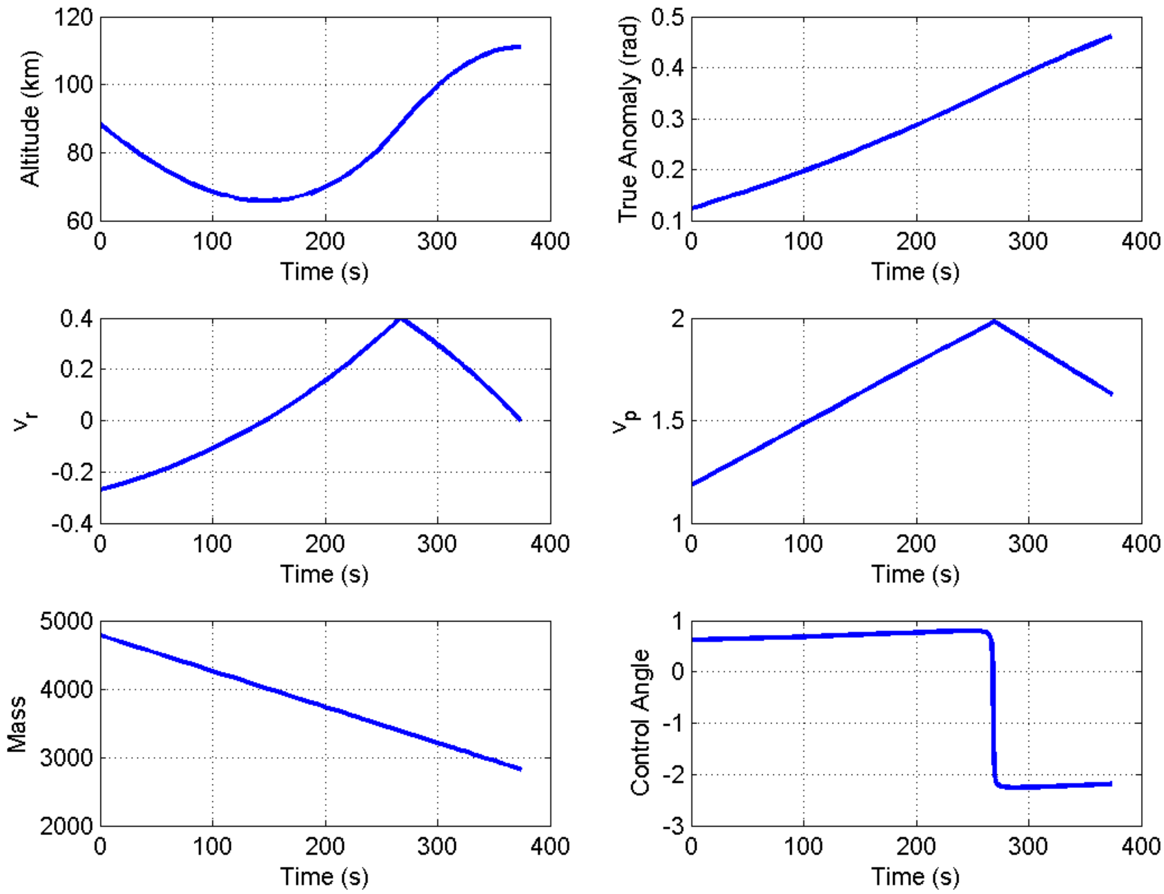

Figure 15: Summarized State and Control Histories for the Pull-Up Abort

We observe that the ascent stage gains altitude while also catching up in the true anomaly. There is a near discontinuity in the control at about 270 seconds. At this point, the LEM switches from maximizing its orbital energy to circularizing its orbit and coasting in to rendezvous with the CSM.

7.5 'Deflection' Problem Formulation

If the abort is initiated some time greater than about 150 seconds into the optimal descent, the CSM will be too far ahead in its orbit for a standard pull-up and phasing maneuver. Instead, I found that the optimal trajectory actually has the LEM fall down toward the moon, gaining orbital velocity at no control cost. The LEM can coast at this high speed orbit until the appropriate phase relationship to the CSM has been restored. I have so named this strategy the Deflection abort since, in MCI coordinates, the trajectory effectively deflects around the Moon before rendezvousing.

The minimization problem for the abort phase is stated as follows. Note the descent phase formulation was presented previously.

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & J = t_f \\ \text{subject to} \quad & \dot{x} = f(x,u) \\ & r > R_M + 11km \\ & (m_{t_{as}} - m_{p_{as}}) \leq m \leq m_{t_{as}} \\ & T : \{0, T_{max}\} \end{aligned} \tag{25}$$

Again, the problem has been formulated assuming the astronaut has selected 'Stage Abort', immediately jettisoning the descent stage. There is a new element/assumption made that the ascent engine, while non throttleable (at $T_{max} = 16.0kN$), is restartable. I.e. the ascent stage can coast for some time - this was done repeatedly on Apollo. Moreover, I have included an additional constraint on r that the altitude cannot dip below 11 km. The motivation for this is that there would be a risk of collision with the terrain below 11 km.

The boundary conditions for the abort phases are given as follows. The abort has been initiated at $t = 280s$. The initial conditions for this phase are that the interior points between optimal descent and abort are continuous:

$$t_f^1 = t_0^2 \tag{26}$$
$$r(t_f^{(1)}) = r(t_0^{(2)}) \tag{27}$$
$$\theta(t_f^{(1)}) = \theta(t_0^{(2)}) \tag{28}$$
$$v_r(t_f^{(1)}) = v_r(t_0^{(2)}) \tag{29}$$
$$v_t(t_f^{(1)}) = v_t(t_0^{(2)}) \tag{30}$$
$$m_0 = m_{t_{as}} \tag{31}$$

$$r_f = r_M + h_p \qquad \theta_f = \frac{v_p}{r} * \Delta t \qquad v_{r_f} = 0 \qquad v_{t_f} = \sqrt{\frac{\mu}{r_f}} \tag{32}$$

The final condition on θ ensures that the ascent stage catches up and rendevous exactly with the CSM.

7.6 'Deflection' Results

The Stage Abort has been initiated after the 'Point of No Return' so a Pull-Up abort cannot be the solution. Instead we observe the predicted behavior shown in Figure 16.

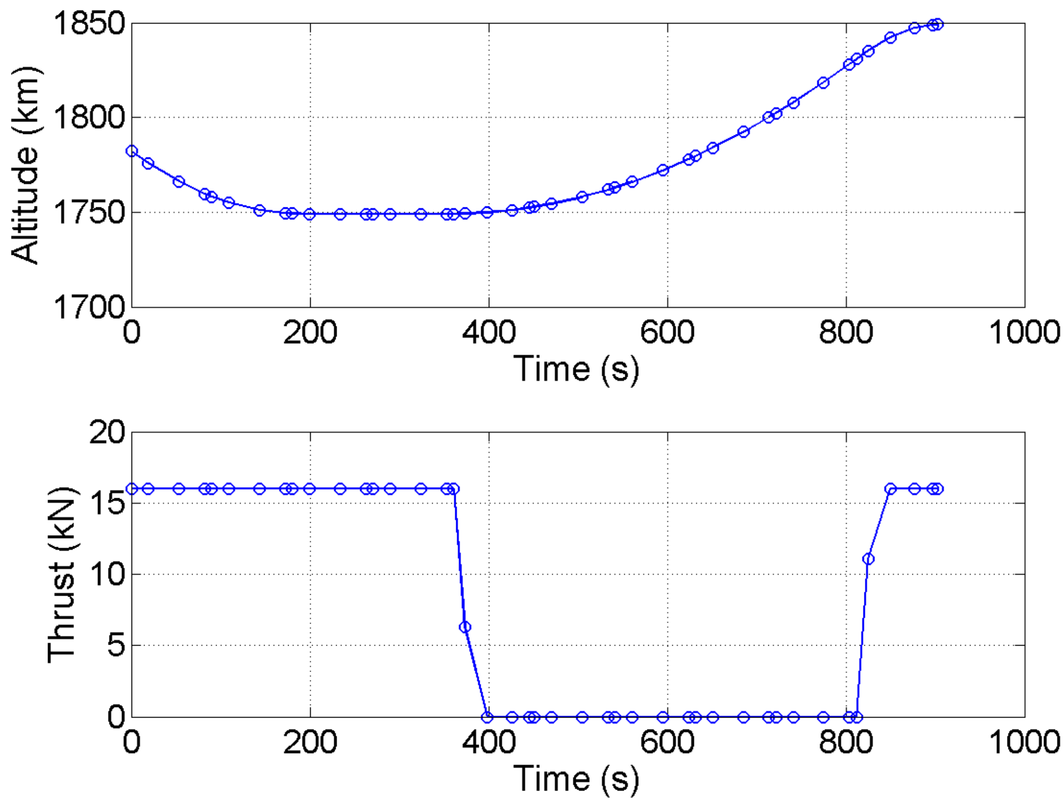

Figure 16: Altitutde Trajectory forthe Deflection Lunar Abort

We observe an engine shutoff coasting period followed by the altitude raising burn for rendezvous. We note that this deflection maneuver burns less fuel than a late pull-up abort or the optimal ascent profile. Although the LEM travels a wider range of altitudes, the orbital velocity is maintained high. Thus the control actually acts to decrease the tangential velocity (thereby increasing the orbital altitude).

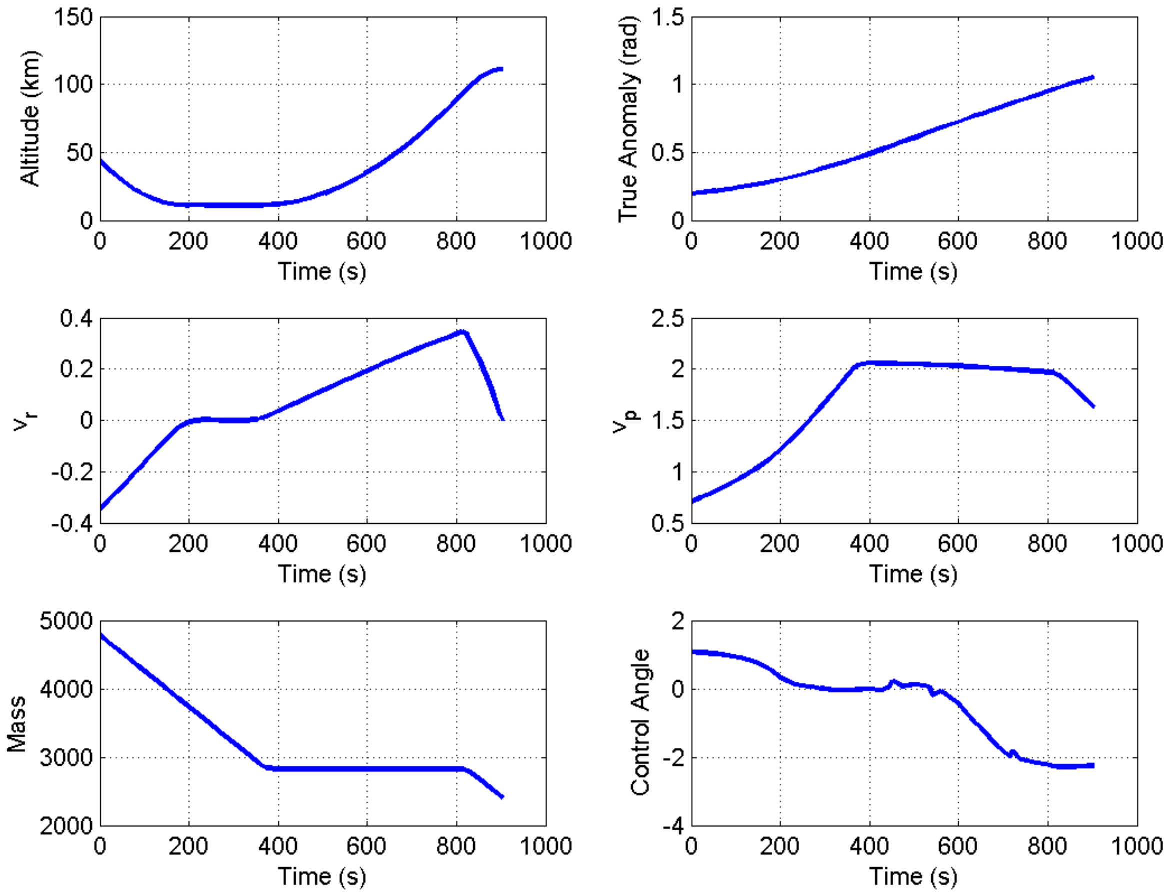

Figure 17: Summarized State and Control Histories for the Deflection Lunar Abort

7.7 'Two-Staged' Problem Formulation

In this subproblem, we consider the case that the astronaut has selected the 'Abort' button but after the so-called 'Point of No Return'. In this case, the descent stage is completely depleted of fuel before jettisoning the stage and completing the rendezvous with just the ascent stage. This is a more fuel efficient strategy than jettisoning immediately. According to [8], this was the preferred abort strategy if a direct 'pull-up' could not be achieved. In this formulation, there are three additional phases after an abort is initiated during descent.

The first phase is the optimal descent portion. The second phase takes the LEM into a low altitude orbit (no less than 11 km) and coasts. The third phase depletes the descent stage fuel by maximizing orbital altitude. The final phases completes the rendezvous with the CSM. Second phase formulation:

$$
\begin{aligned}
\underset{u}{\text{minimize}} \quad & J = t_f \\
\text{subject to} \quad & \dot{x} = f(x,u) \\
& r > R_M + 11km \\
& (m_{t_{as}} + m_{t_{ds}} - m_{p_{ds}}) \leq m \leq m_{t_{ds}} + m_{t_{as}} \\
& T : \{0, [0.1T_{max}, 0.6T_{max}], T_{max}\}
\end{aligned} \tag{33}
$$

The boundary conditions for this phase is given as follows. The abort has been initiated at $t = 180s$. The initial conditions for this phase are that the interior points between optimal descent and abort are continuous:

$$t_f^1 = t_0^2 \tag{34}$$
$$r(t_f^{(1)}) = r(t_0^{(2)}) \tag{35}$$
$$\theta(t_f^{(1)}) = \theta(t_0^{(2)}) \tag{36}$$
$$v_r(t_f^{(1)}) = v_r(t_0^{(2)}) \tag{37}$$
$$v_t(t_f^{(1)}) = v_t(t_0^{(2)}) \tag{38}$$
$$m(t_f^{(1)}) = m(t_0^{(2)}) \tag{39}$$

$$r_f = r_M + 11 \qquad v_{r_f} = 0 \qquad v_{t_f} = \sqrt{\frac{\mu}{r_f}} \tag{40}$$

We note that this is a stable orbit and the LEM can coast until the appropriate positioning relative to the CSM has been made. The third phase maximizes the orbital altitude. We include a requirement that the descent stage be jettisoned while in a circular orbit. In this way, any anomalies for ascent stage restart keep the astronauts in a homeostatic orbit (not a ballistic trajectory toward the surface).

$$
\begin{aligned}
\underset{u}{\text{minimize}} \quad & J = -r_f \\
\text{subject to} \quad & \dot{x} = f(x,u) \\
& r > R_M + 11km \\
& (m_{t_{as}} + m_{t_{ds}} - m_{p_{ds}}) \leq m \leq m_{t_{ds}} + m_{t_{as}} \\
& T : \{0, [0.1T_{max}, 0.6T_{max}], T_{max}\}
\end{aligned} \tag{41}
$$

The boundary conditions for this phase is given as follows.

$$t_f^{(2)} = t_0^{(3)} \tag{42}$$
$$r(t_f^{(2)}) = r(t_0^{(3)}) \tag{43}$$
$$\theta(t_f^{(2)}) = \theta(t_0^{(3)}) \tag{44}$$
$$v_r(t_f^{(2)}) = v_r(t_0^{(3)}) \tag{45}$$
$$v_t(t_f^{(2)}) = v_t(t_0^{(3)}) \tag{46}$$
$$m(t_f^{(2)}) = m(t_f^{(3)}) \tag{47}$$

$$v_{r_f} = 0 \qquad v_{t_f} = \sqrt{\frac{\mu}{r_f}} \tag{48}$$

The final phase ensures rendezvous with the CSM and has a familiar formulation.

$$\begin{aligned}
\underset{u}{\text{minimize}} \quad & J = t_f \\
\text{subject to} \quad & \dot{x} = f(x,u) \\
& r > R_M + 11km \\
& (m_{t_{as}} + m_{t_{ds}} - m_{p_{ds}}) \leq m \leq m_{t_{ds}} + m_{t_{as}} \\
& T : \{0, T_{max}\}
\end{aligned} \tag{49}$$

The boundary conditions for this phase is given as follows.

$$t_f^{(3)} = t_0^{(4)} \tag{50}$$
$$r(t_f^{(3)}) = r(t_0^{(4)}) \tag{51}$$
$$\theta(t_f^{(3)}) = \theta(t_0^{(4)}) \tag{52}$$
$$v_r(t_f^{(3)}) = v_r(t_0^{(4)}) \tag{53}$$
$$v_t(t_f^{(3)}) = v_t(t_0^{(4)}) \tag{54}$$
$$m(t_0^{(4)}) = m_{t_{as}} \tag{55}$$

$$r_f = r_M + 111 \qquad v_{r_f} = 0 \qquad v_{t_f} = \sqrt{\frac{\mu}{r_f}} \tag{56}$$

7.8 'Two-Staged' Results

It turns out that the two-staged abort behaves very similar to the Deflection abort, except that the fuel margin on the ascent stage is significantly higher for this two-staged maneuver. However, both strategies are capable of CSM rendezvous after the 'Point of No Return'.

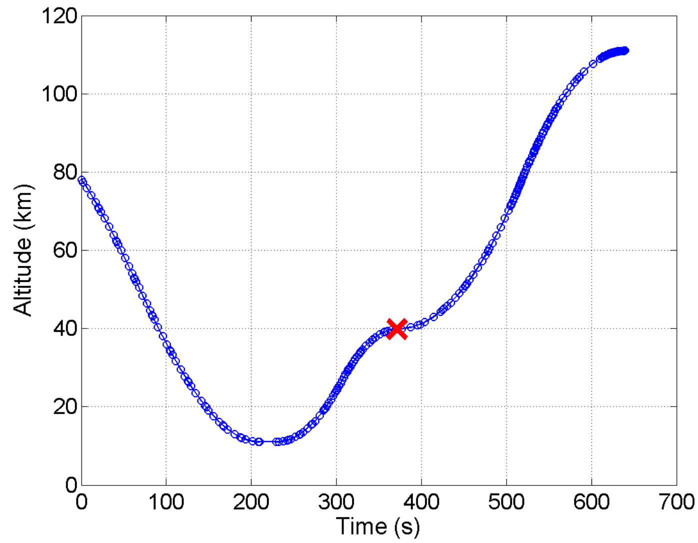

Figure 18: Altitude Trajectory for the Two-Staged Lunar Abort

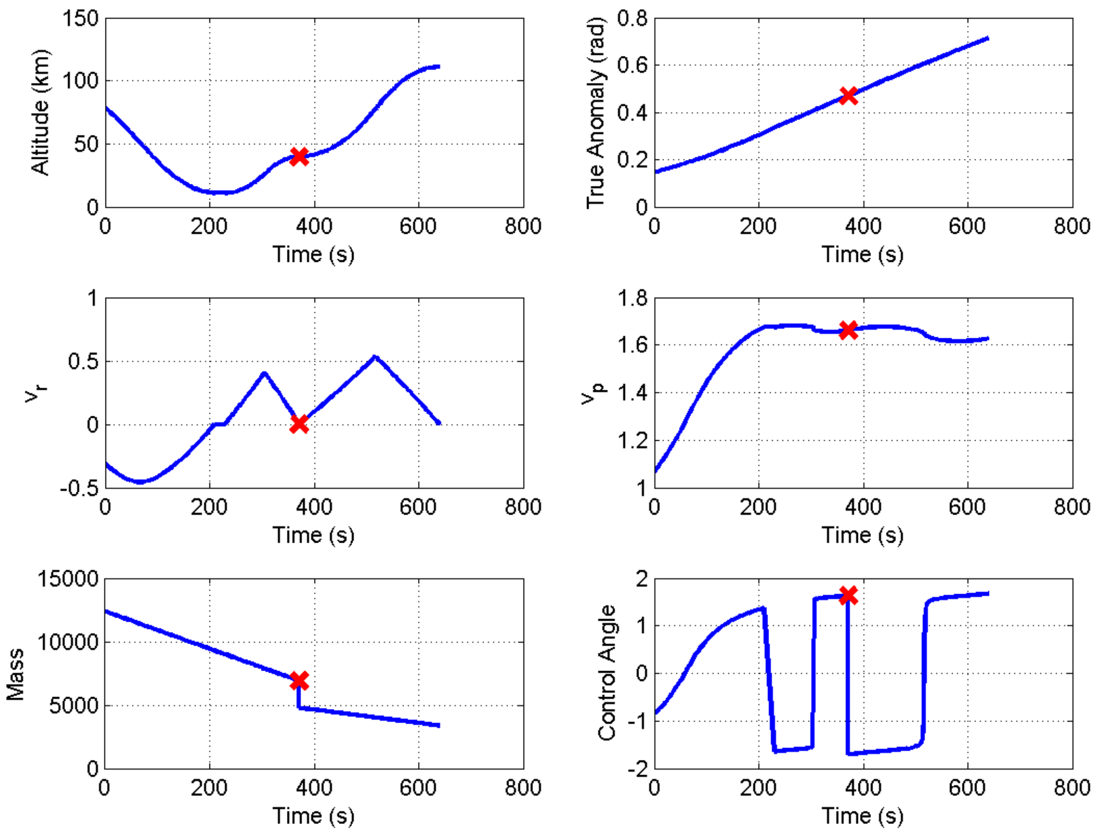

Figure 19: Summarized State and Control Histories for the Two-Staged Lunar Abort

We observe that the trajectory is exactly as expected. The altitude reaches a minimum of 11 km just above the terrain line and can coast as long as needed to maintain the proper phase relationship with the CSM before ascending. Then it reaches as high of a plateau as possible with the descent stage before completing the final rendezvous with just the ascent stage. The final mass margin is significantly higher than the Deflection maneuver.

7.9 'Hop' Problem Formulation

The final abort considered is a hop from the lunar surface. Just as I have defined a 'Point of No Return' for a pull-up, there is a dual point in descent (approximately 11 km) where the astronauts must commit to touchdown. Any abort below 11km in altitude must land before rendezvousing. This reduces the risk of terrain collisions, stage separation anomalies, and surface impact. This case differs from the standard optimal ascent because the CSM can be anywhere in the orbit– i.e. there is a large θ offset that must be accounted for.

Defining optimality conditions for this case is not straightforward. The idea is that if the Hop abort is initiated immediately at touchdown, up until about 25 minutes after touchdown, the optimal strategy would be to ascend to a minimum altitude orbit until the appropriate relative phasing is achieved and ascent begins. If the Hop abort is initiated sometime after this critical time (approximately greater than 25 minutes), the optimal strategy would be to boost to a high altitude orbit, reversing the CSM/LEM roles so that the CSM is the one catching up from behind.

As will be presented, I was unable to converge to an optimal solution without the 'Deus Ex Machina' condition, due to insufficient mass margins. The Deus Ex Machina condition is borrowed from classical literature, when the hand of God is required to conveniently solve the problem. The presented solution was formulated as follows.

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & J = t_f \\ \text{subject to} \quad & \dot{x} = f(x,u) \\ & r > R_M \\ & (m_{t_{as}} - m_{p_{as}}) \leq m \leq m_{t_{as}} \\ & T : \{0, T_{max}\} \end{aligned} \tag{57}$$

Again, the engine is non-throttleable so the thrust is limited to either off or full throttle ($T_{max} = 16.0kN$). The boundary conditions are given as follows. The initial conditions are:

$$r_0 = r_M \qquad \theta_0 = 0 \qquad v_{r_0} = 0 \qquad v_{t_0} = 0 \qquad m_0 = m_{t_{as}} \tag{58}$$

$$r_f = r_M + h_p \pm 30km \qquad v_{r_f} = 0 \qquad v_{t_f} = \sqrt{\frac{\mu}{r_f}} \tag{59}$$

We have chosen to put the LEM in an orbit that has a 30km altitude differential to the parking orbit. This is exactly what was done in the actual Apollo flight profile. In this way, CSM rendezvous is guaranteed in less than 3 hours (about 2 orbits) [10]. The second phase, in theory, propagates the orbit from the first stage until the CSM is in position before ascending/descending.

$$t_f^{(1)} = t_0^{(2)} \tag{60}$$
$$r(t_f^{(1)}) = r(t_0^{(2)}) \tag{61}$$
$$\theta(t_f^{(1)}) = \theta(t_0^{(2)}) \tag{62}$$
$$v_r(t_f^{(1)}) = v_r(t_0^{(2)}) \tag{63}$$
$$v_t(t_f^{(1)}) = v_t(t_0^{(2)}) \tag{64}$$
$$m(t_0^{(1)}) = m(t_0^{(2)}) \tag{65}$$

$$r_f = r_M + h_p \qquad \theta_f = \frac{v_p}{r} * \Delta t \qquad v_{r_f} = 0 \qquad v_{t_f} = \sqrt{\frac{\mu}{r_f}} \tag{66}$$

7.10 'Hop' Results

As described in both [10] [4], this strategy was to be implemented by the LEM for an equivalent 3 phase orbit insertion hop. My results showed that, given the fixed mass fraction of the ascent stage, such a hop maneuver was infeasible without the Deus Ex Machina condition. Deus Ex Machina invokes the CSM to perform phasing maneuvers to rendezvous with the ascent stage if the LEM is fuel depleted. As shown in the Appendix plots Figure 25, the fuel required by the LEM to first park in an orbit, and then re-rendezvous was simply too much.

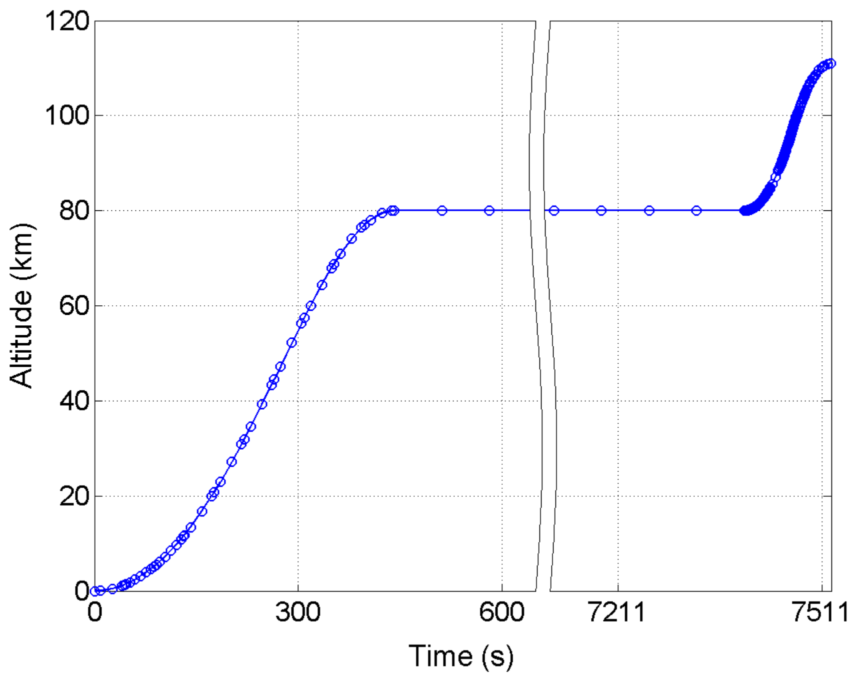

Figure 20: Altitude Trajectory for the Hop and Catch-Up Lunar Abort

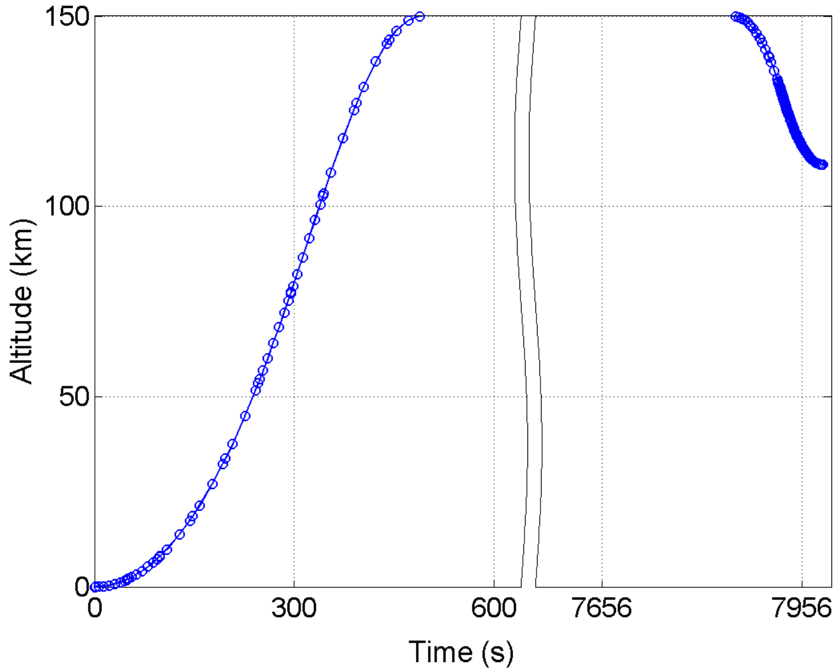

Figure 21: Altitude Trajectory for the Hop and Wait Lunar Abort

We note that the key difference in analysis between the work presented here, and the actual Apollo flight profile is that here we use circular parking orbits. As an extension to this work, the author also considered and solved this problem using elliptical parking orbits (exactly those proposed by the Apollo flight profile), but was unable to show a marked performance improvement to close the problem. As a final note, we see that the fuel difference between an 80km parking orbit and a 150km parking orbit is very small. Although 80km is much lower in altitude, the orbital speed is higher, so the delta-v required is not that much less than for a 150km orbit.

8 Summary and Conclusions

In this report, we have investigated the moon landing of the Apollo Lunar Lander. Specifically, we have simulated the optimal trajectory and proposed control inputs for three optimal control problems: descent, ascent and abort. Due to the complexity of the abort problem, and nature of the physical two-stage system and three-body problem, no single problem formulation was sufficient to emcompass the solution to the optimal abort. Instead I proposed a solution set that guarantees optimal CSM rendezvous from any abort during the mission.

The actual Apollo flight profile was not optimal in time or mass. Instead, the Apollo mission accounted for many of the simplifications I made for my analysis. Future work on this project would address some systematic errors neglected here for simplicity.

- *Risk Weighted Objective Function*: The best objective function may not be final time or final mass. Instead a risk weighted function of time and fuel and other variables (such as terrain, dispersions, astronaut state of health, window field of view) may be chosen instead. I would expect optimality in this sense to converge on a strategy that is much more similar to the actual Apollo flight profile.

- *Sensor Noise*: The optimal solution presented here assumed perfect state knowledge from the guidance instruments and landing radar. In reality, these introduced large dispersions in the landing zone and rendezvous coordinates. In the future, this could be modeled as gaussian stochastic (or biased) variations in the state and environment.
- *Actuator Modeling*: The ascent and descent propulsion engines were assumed to be ideal nozzles with constant thrust parameters and infnite frequency control bandwidth. Other non-negligible dynamics include engine shutdown and jettison time constants.
- *Environmental Disturbances*: We have considered the Moon as the only celestial body within the sphere of influence of this problem. However, in the same way that the Moon affects the tides on the Earth, I expect a similar effect of the Earth on the Moon. Although the Moon does not have an atmosphere, there still exists spaceweather disturbances such as solar winds (and coronal mass ejections).
- *Time Lag*: We have also assumed infinite processing speed and instantaneous ground communications. Unfortunately, this is unrealistic. The actual Apollo trajectory needed to account for crew communication periods, as well as loss of signal on the dark side of the moon (where orbital maneuvers were usually not to be performed).
- *Other Unmodeled Physics*: This report also does not address the Reaction Control System (which can affect trajectories), fluid slosh, or other electromechanical latencies.

$\mathbb{GPOPS} - \mathbb{II}$ was used as the nonlinear solver and I was grateful for its versatility and inuitive Matlab interface. When provided a sufficiently constrained problem with a reasonable initial guess and no discontinuities, $\mathbb{GPOPS} - \mathbb{II}$ was able to converge to an optimal solution very quickly (within seconds). I noticed that if $\mathbb{GPOPS} - \mathbb{II}$ struggled to meet the solution tolerance, I was able to tune the problem formulation to achieve convergence. This was done by adjusting the maximum number of iterations and collocation points, recasting the boundary conditions or state limits, and iteratively using the previous solution for the current best guess. In the end, I was impressed by the simplicity of $\mathbb{GPOPS} - \mathbb{II}$ and intend to use it in future optimal control problems throughout my career.

Since Apollo, particularly in the 21st century, several commercial and governmental entities have exhibited high interest in autonomously piloted lunar missions. For example, the Google Lunar X Prize is incentivizing privately-funded spaceflight teams to design a robot to safetly land on the moon. Morover, a manned lunar base has been cited in NASA's cross enterprise roadmap documents for upcoming interplanetary missions. The methodology presented in this paper may be scaled to such manned or unmanned missions to the Moon, Mars and beyond.

References

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Appendices

A Visualization

A custom simulation was developed for this project to show the audience how the trajectory progress through time. The custom GUI designed is shown in Figure 22

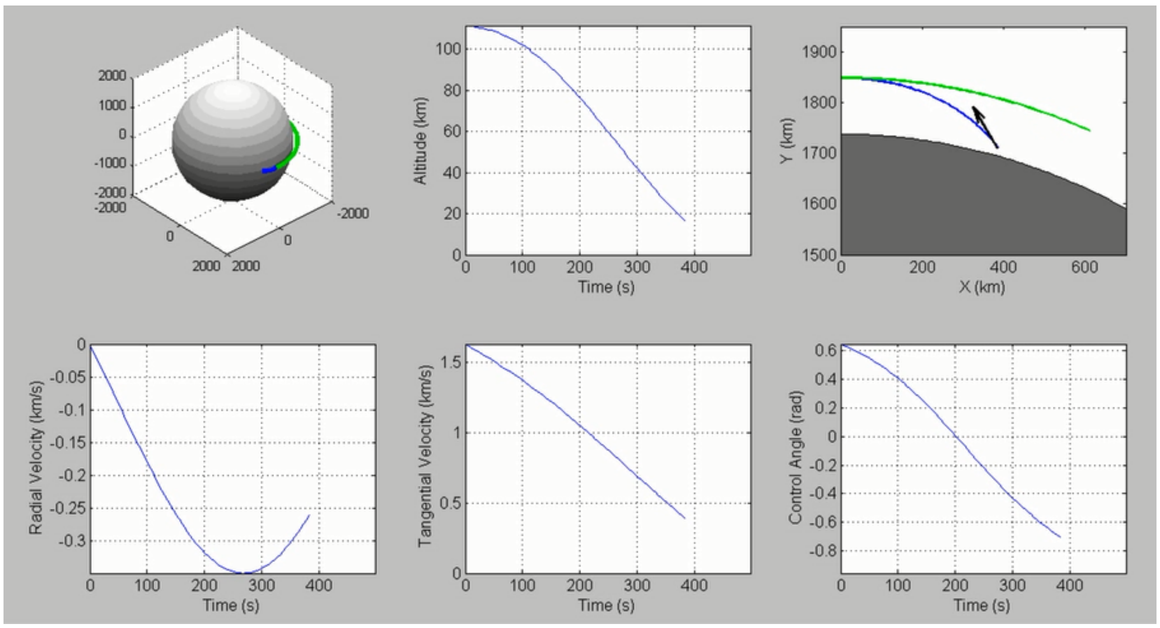

Figure 22: Custom Visualization for Real-Time Plotting

Duncan Miller, *16.323 Student*
duncanlm@mit.edu

B Additional Figures

Additional individual figures per state have been generated and populated in the published html report located in the html subfolder.

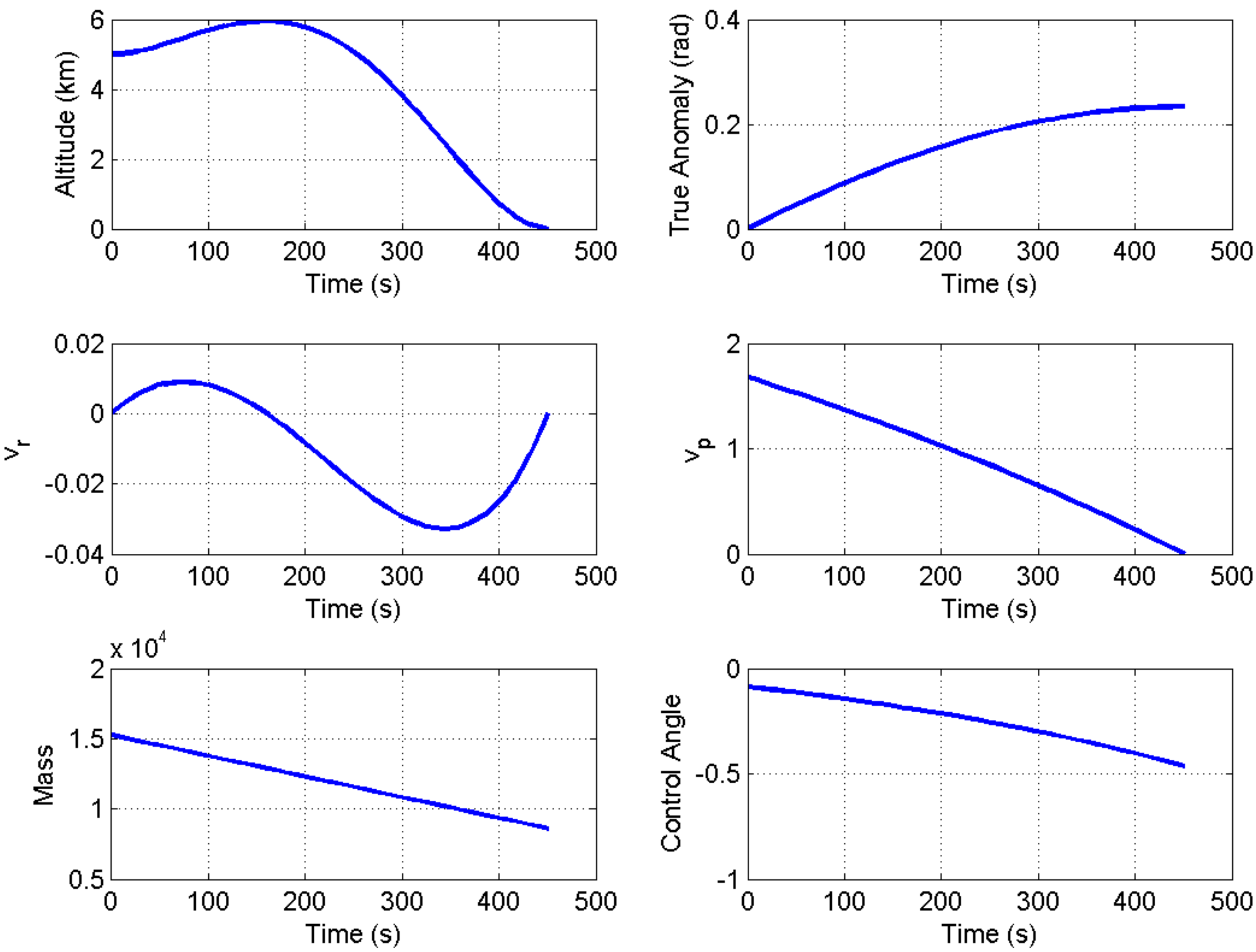

Figure 23: State and Control Histories for Lunar Descent from 5km

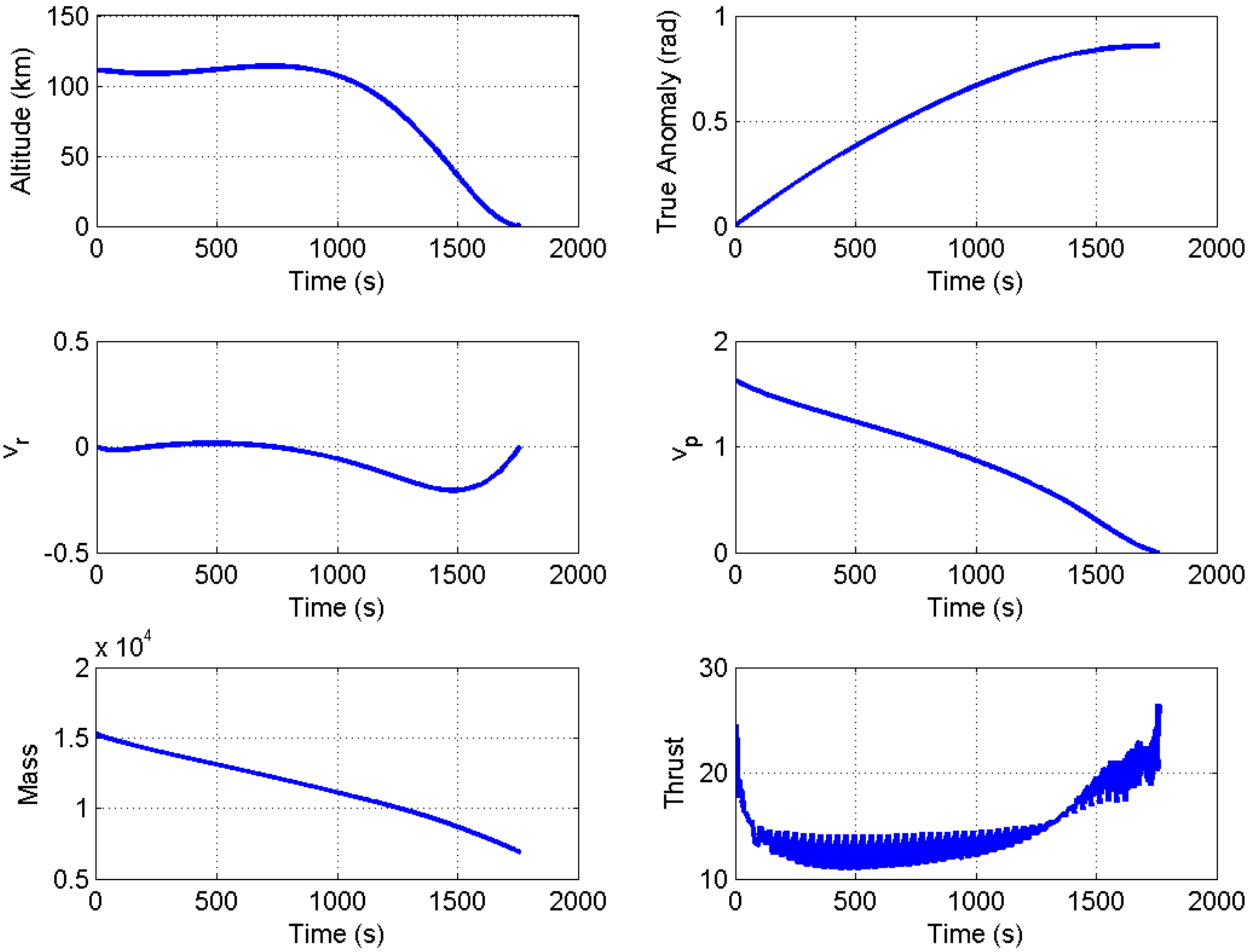

Figure 24: State and Control Histories Minimum Mass Lunar Descent

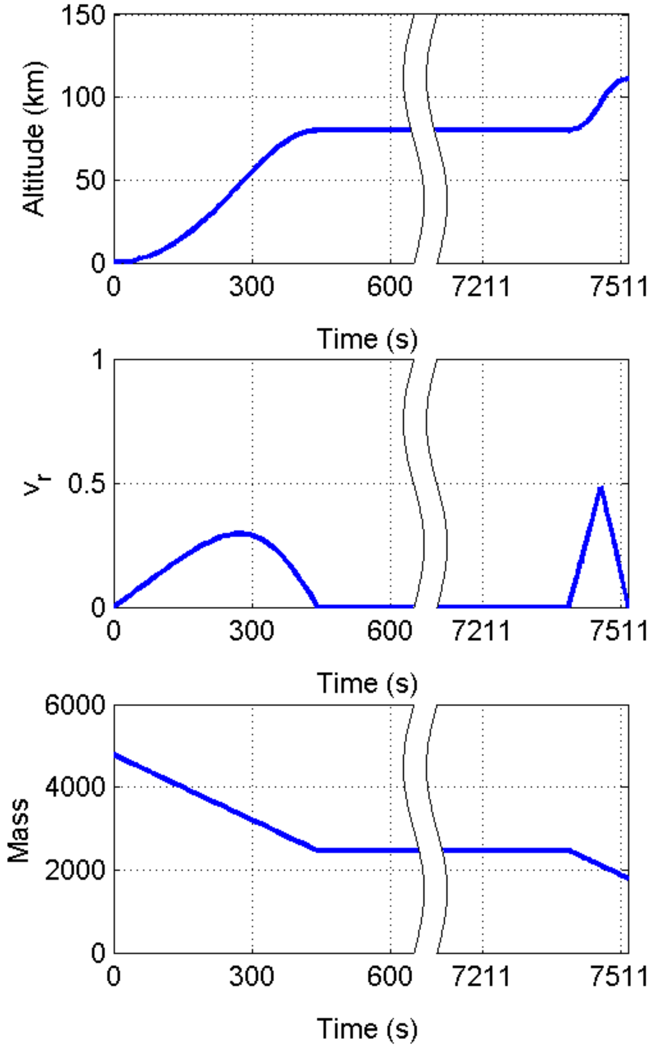

Figure 25: Summarized State and Control Histories for the Hop and Catch Up Lunar Abort

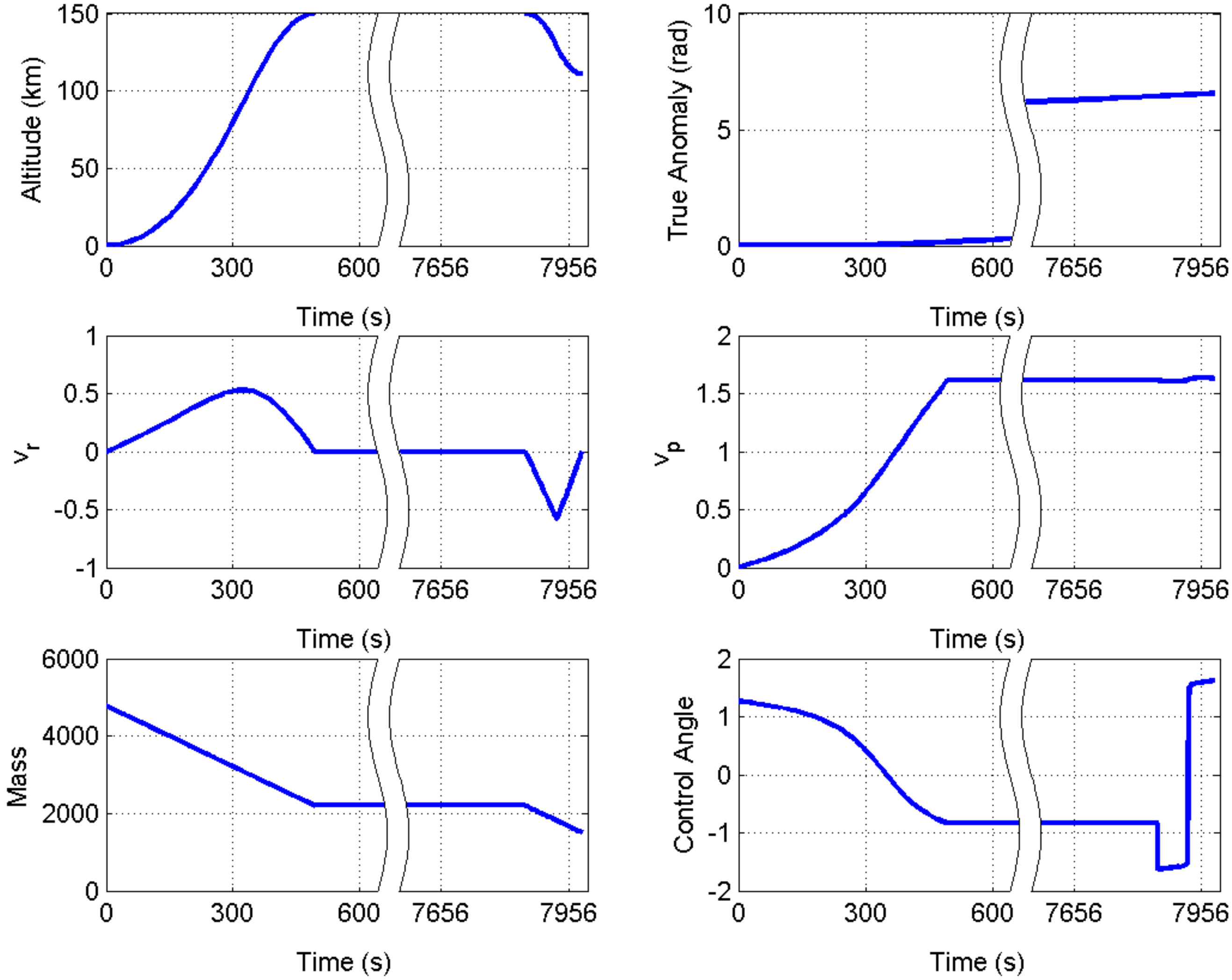

Figure 26: Summarized State and Control Histories for the Hop and Wait Lunar Abort